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Six-vertex model with domain wall boundary conditions: variable inhomogeneities

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Abstract

We consider the six-vertex model with domain wall boundary conditions. We choose the inhomogeneities as solutions of the Bethe Ansatz equations. These equations have many solutions, so we can consider a wide variety of inhomogeneities. For certain choices of the inhomogeneities we study arrow correlation functions on the horizontal line going through the centre. In particular, we obtain a multiple integral representation for the emptiness formation probability that generalizes the known formulae for *XXZ* antiferromagnets.

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1. Introduction

The six-vertex model was first introduced in [1]. It was solved exactly by Lieb [2] and Sutherland [3] in 1967 by means of a Bethe Ansatz for periodic boundary conditions. Later the six-vertex model was also studied in the presence of several other boundary conditions [4–6]. Domain wall boundary conditions were introduced in 1982 [7]. These boundary conditions are interesting because they allow for the derivation of determinant representations for correlation functions [8]. It was realized recently that the six-vertex model in the presence of such boundary conditions is extremely helpful in the enumeration of alternating sign matrices [9, 10]. The bulk free energy for these boundary conditions was calculated in [11].

In this paper we show that for special choices of inhomogeneities, one can compute the free energy and some correlation functions of the system with domain wall boundaries. This observation might be useful because we expect some properties of the model to be independent of the inhomogeneities, i.e. to depend only on the anisotropy parameter. In the simplest situation, the correlation functions coincide with those for periodic boundary conditions. We will continue the line of research of [12] and will primarily be interested in the emptiness formation probability (EFP) which was first introduced in [13]. In the latter paper a multiple integral expression for the EFP was obtained for the first time for the ground state of the *XXX* antiferromagnet. Here we will consider the EFP for more general Bethe states and will generalize the multiple integral expression of these cases.

2. Quantum inverse scattering

We consider the inhomogeneous six-vertex model with domain wall boundary conditions. It is defined as the six-vertex model on a $M \times M$ square lattice with fixed boundary conditions: arrows on the horizontal (vertical) edges are outgoing (ingoing). Furthermore, spectral parameters λ_i and μ_k are attached to line i and column k. We choose the following parameterization of the usual Boltzmann weights a, b and c:

$$\begin{aligned} a(\lambda) &= 1 \\ b(\lambda) &= \frac{\sinh(\lambda - \eta/2)}{\sinh(\lambda + \eta/2)} \\ c(\lambda) &= \frac{\sinh \eta}{\sinh(\lambda + \eta/2)}. \end{aligned} \tag{1}$$

We want to make use of the formalism of the algebraic Bethe Ansatz. For this purpose the Boltzmann weights are collected in a matrix L,

$$L(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

The monodromy matrix is then defined as an ordered product of the L-operators,

$$T(\lambda; \{\mu_k\}) = L(\lambda - \mu_M) \ldots L(\lambda - \mu_2) L(\lambda - \mu_1) \tag{3}$$

where $L(\lambda - \mu_k)$ acts on the kth factor of the physical space $\mathbb{C}^{2^M}$, and the auxiliary space $\mathbb{C}^2$. For clarity, we will in the following often suppress the explicit dependence of T on $\{\mu_k\}$. As an operator on the two-dimensional auxiliary space, $T(\lambda)$ can be written as

$$T(\lambda) = \begin{pmatrix} \mathbf{A}(\lambda) & \mathbf{B}(\lambda) \\ \mathbf{C}(\lambda) & \mathbf{D}(\lambda) \end{pmatrix} \tag{4}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are operators on the physical space $\mathbb{C}^{2^M}$. The trace of the monodromy operator,

$$\mathbf{T}(\lambda) = \mathbf{A}(\lambda) + \mathbf{D}(\lambda) \tag{5}$$

is the usual transfer matrix corresponding to the model with periodic boundary conditions. Because of the parameterization (1), the monodromy matrix $T(\lambda)$ satisfies the following intertwining relation:

$$\check{R}(\lambda - \mu)\,[T(\lambda) \otimes T(\mu)] = [T(\lambda) \otimes T(\mu)]\,\check{R}(\lambda - \mu). \tag{6}$$

The R-matrix $\check{R}$ is defined by

$$\check{R}(\lambda) = P L(\lambda + \eta/2) \tag{7}$$

where P is the permutation operator on the space $\mathbb{C}^2 \otimes \mathbb{C}^2$. Equation (6) embodies several commutation relations between the operators $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ defined in (4).

The fixed boundary conditions imply the following formal expression for the partition function [7]:

$$Z_M(\{\lambda_i\}, \{\mu_k\}) = \langle\Downarrow|\mathbf{B}(\lambda_1; \{\mu_k\}) \ldots \mathbf{B}(\lambda_M; \{\mu_k\})|\Uparrow\rangle \tag{8}$$

where $|\Uparrow\rangle(|\Downarrow\rangle)$ is the state with all spins up (down). In this paper we shall be interested only in the case where M is even and the $\{\lambda_i\}$ are chosen as $\{\lambda_i\}_{i=N+1}^{M} = \{\lambda_i\}_{i=1}^{N}$ for $i = 1, \ldots, N$, where $N = M/2$. This allows us to rewrite the partition function in a convenient way [12]. First define the states

$$|N\rangle = \mathbf{B}(\lambda_1) \ldots \mathbf{B}(\lambda_N)|\Uparrow\rangle \qquad \langle N| = \langle\Uparrow|\mathbf{C}(\lambda_1) \ldots \mathbf{C}(\lambda_N). \tag{9}$$

Let $\mathbf{R} = \prod_{k=1}^{M} \sigma_k^x$ be the flip operator on the physical space that flips all arrows. We then find that

$$Z_M(\{\lambda_i\}, \{\mu_k\}) = \langle N|\mathbf{R}|N\rangle. \tag{10}$$

Formal expressions for correlation functions can also be written concisely in this notation. For example, the probability that all arrows located at the columns $k_1, \ldots, k_n$ and between the lines N and $N + 1$ are down is given by

$$\left\langle\pi_{k_1} \ldots \pi_{k_n}\right\rangle = \frac{\left\langle N|\mathbf{R}\pi_{k_1} \ldots \pi_{k_n}|N\right\rangle}{\langle N|\mathbf{R}|N\rangle} \tag{11}$$

where $\pi_k = \frac{1}{2}(1-\sigma_k^z)$. Averages like (11) can be calculated using the solution of the quantum inverse scattering problem for the operators π_k [14, 15],

$$\pi_k = \prod_{l=1}^{k-1} \mathbf{T}(\mu_l + \eta/2)\, \mathbf{D}(\mu_k + \eta/2) \prod_{l=k+1}^{M} \mathbf{T}(\mu_l + \eta/2). \tag{12}$$

From this expression it is clear that the correlation function (11) simplifies when the k_i are nearest neighbours.

3. Bethe Ansatz

A marvellous aspect of formulae (11) and (12) is that the set of inhomogeneities can be chosen such that the state $|N\rangle$ is an eigenstate of $\mathbf{T}$. In fact, there are many choices possible that have this property. In this section we will derive explicit expression for the correlation function (11) corresponding to such choices. We will closely follow a similar derivation given in [16] for one particular choice of the inhomogeneities, namely those corresponding to the ground state of the antiferromagnetic XXZ quantum spin chain. To begin, we fix the set of inhomogeneities $\{\lambda_i\}$ to be a solution of the Bethe Ansatz equations,

$$\frac{a(\lambda_j)}{d(\lambda_j)} \prod_{\substack{k=1\\k\neq j}}^{N} \frac{b(\lambda_j - \lambda_k + \eta/2)}{b(\lambda_k - \lambda_j + \eta/2)} = 1 \qquad 1 \leqslant j \leqslant N \tag{13}$$

where $a(\lambda) = 1$ and $d(\lambda) = \prod_{k=1}^{M} b(\lambda - \mu_k)$ are the eigenvalues of operators $\mathbf{A}(\lambda)$ and $\mathbf{D}(\lambda)$ respectively on the reference state $|\Uparrow\rangle$. Note that these equations imply that $\prod_{j=1}^{N} d(\lambda_j) = 1$. It will be useful to rewrite the Bethe Ansatz equations in their logarithmic form,

$$\varphi(\lambda_j) = \pi \pmod{2\pi} \qquad 1 \leqslant j \leqslant N \tag{14}$$

where the function φ is defined by

$$\begin{aligned}\varphi(\lambda) &= -\mathrm{i}\ln\frac{a(\lambda)}{d(\lambda)} - \mathrm{i}\sum_{k=1}^{N}\ln\frac{b(\lambda-\lambda_k+\eta/2)}{b(\lambda_k-\lambda+\eta/2)} \\ &= \mathrm{i}\ln\frac{d(\lambda)}{a(\lambda)} + \mathrm{i}\sum_{k=1}^{N}\ln\left(-\frac{\sinh(\eta+\lambda-\lambda_k)}{\sinh(\eta-\lambda+\lambda_k)}\right).\end{aligned} \tag{15}$$

If the $\{\lambda_i\}$ are a solution of the Bethe Ansatz equations, the state $|N\rangle$ is a common eigenstate of $\mathbf{R}$ with eigenvalue ± 1 and of $\mathbf{T}(\lambda)$ with eigenvalue $t(\lambda)$ given by

$$\begin{aligned}t(\lambda) &= a(\lambda)\prod_{i=1}^{N} b^{-1}(\lambda_i-\lambda+\eta/2) + d(\lambda)\prod_{i=1}^{N} b^{-1}(\lambda-\lambda_i+\eta/2) \\ &= a(\lambda)\left(1+\mathrm{e}^{-\mathrm{i}\varphi(\lambda)}\right)\prod_{i=1}^{N} b^{-1}(\lambda_i-\lambda+\eta/2).\end{aligned} \tag{16}$$

If $\{\lambda_i\}$ obey the Bethe Ansatz equations (13) and $\{\xi_j\}$ are a set of parameters, then the following holds [16, 17]:

$$\langle\Uparrow|\prod_{j=1}^{N}\mathbf{C}(\xi_j)\prod_{i=1}^{N}\mathbf{B}(\lambda_j)|\Uparrow\rangle = \frac{\det t'}{\det V} \tag{17}$$

where

$$t'_{ij} = \frac{\partial t(\xi_i)}{\partial\lambda_j} \qquad V_{ij} = \frac{1}{\sinh(\xi_i-\lambda_j)}. \tag{18}$$

A useful formula that we will use in the following is

$$\det V\prod_{k,l=1}^{N}\sinh(\xi_k-\lambda_l) = \prod_{\substack{k,l=1\\k<l}}^{N}\sinh(\lambda_k-\lambda_l)\,\sinh(\xi_l-\xi_k). \tag{19}$$

From (10) it follows that the partition sum is given by the norm of the Bethe state $|N\rangle$. An expression for the norm of a Bethe state in terms of a determinant is given by the following formula which may be obtained by specializing the $\{\xi_j\}$ in (17) to $\{\lambda_i\}$:

$$\begin{aligned}\langle N|N\rangle &= \langle\Uparrow|\prod_{i=1}^{N}\mathbf{C}(\lambda_i)\prod_{i=1}^{N}\mathbf{B}(\lambda_i)|\Uparrow\rangle \\ &= \sinh(\eta)^N\left(\prod_{\substack{i,j=1\\i\neq j}}^{N}\frac{\sinh(\lambda_i-\lambda_j+\eta)}{\sinh(\lambda_i-\lambda_j)}\right)\det\varphi'\end{aligned} \tag{20}$$

where

$$\varphi'_{ij} = -\mathrm{i}\left(\frac{\partial\varphi(\lambda)}{\partial\lambda_j} + \delta_{ij}\frac{\partial\varphi(\lambda)}{\partial\lambda}\right)\bigg|_{\lambda=\lambda_i}. \tag{21}$$

The determinant formula for the norm of a Bethe wave function was first conjectured by Gaudin [18]. Due to the complicated nature of the Bethe wave function a proof was not available till the development of the quantum inverse scattering method. The first proof of the determinant formula of the norm of the Bethe wave function for the *XXZ* spin chain was given by Korepin [7].

More work has to be done to obtain an expression for the correlation function (11). Here we will treat the case when the k_i are nearest neighbours. Using the solution for the quantum inverse scattering (12) and the fact that $|N\rangle$ is an eigenstate of $\mathbf{T}(\lambda)$ and $\mathbf{R}$ one finds

$$\langle \pi_{k+1} \ldots \pi_{k+n} \rangle = \prod_{j=1}^{n} t^{-1}(\mu_{k+j} + \eta/2) \frac{\langle N| \prod_{j=1}^{n} \mathbf{D}(\mu_{k+j} + \eta/2)|N\rangle}{\langle N|N\rangle}. \tag{22}$$

Since $d(\mu_k + \eta/2) = 0$, the inverse eigenvalue $t^{-1}(\mu_k + \eta/2)$ takes the simple form

$$t^{-1}(\mu_k + \eta/2) = \prod_{i=1}^{N} \frac{\sinh(\lambda_i - \mu_k - \eta/2)}{\sinh(\lambda_i - \mu_k + \eta/2)}. \tag{23}$$

The action of a product of the operators $\mathbf{D}$ on $|N\rangle$ can be calculated and is given by

$$\begin{aligned}
&\prod_{j=1}^{n} \mathbf{D}(\lambda_{N+j}) \prod_{k=1}^{N} \mathbf{B}(\lambda_k)|\uparrow\rangle \\
&\quad = \sum_{i_1=1}^{N+1} \sum_{\substack{i_2=1\\ i_2\neq i_1}}^{N+2} \cdots \sum_{\substack{i_n=1\\ i_n\neq i_1,\ldots,i_{n-1}}}^{N+n} G_{i_1,\ldots,i_n}\left(\{\lambda_i\}_{i=1}^{N+n}\right) \prod_{\substack{k=1\\ k\neq i_1,\ldots,i_n}}^{N+n} \mathbf{B}(\lambda_k)|\uparrow\rangle
\end{aligned} \tag{24}$$

where the function G is given by

$$G_{i_1,\ldots,i_n}\left(\{\lambda_i\}_{i=1}^{N+n}\right) = \prod_{l=1}^{n} d\left(\lambda_{i_l}\right) c\big(\lambda_{i_l} - \lambda_{N+l} + \eta/2\big) \prod_{\substack{k=1\\ k\neq i_1,\ldots,i_l}}^{N+l} b^{-1}\big(\lambda_{i_l} - \lambda_k + \eta/2\big). \tag{25}$$

We will set $\lambda_{N+j} = \mu_{k+j} + \eta/2$ to calculate the n-point correlation function (22). Since $d(\mu_k + \eta/2) = 0$ this means that the sums in (24) only run up to $i_l = N$.

From (24) it is seen that we need to calculate the scalar products of the type

$$S(\{\lambda_i\}, \{\lambda_1, \ldots, \lambda_{N-n}, \mu_{k+1}, \ldots, \mu_{k+n}\}) = \frac{\langle N| \prod_{i=1}^{N-n} \mathbf{B}(\lambda_i) \prod_{j=1}^{n} \mathbf{B}(\mu_{k+j} + \eta/2)|\uparrow\rangle}{\langle N|N\rangle}. \tag{26}$$

Using (17) and (19) one may express S as a ratio of determinants,

$$\begin{aligned}
S(\{\lambda_i\}, \{\lambda_1, \ldots, \lambda_{N-n}, \mu_{k+1}, \ldots, \mu_{k+n}\}) &= \prod_{\substack{i,j=1\\ i<j}}^{n} \frac{\sinh(\lambda_{N-n+j} - \lambda_{N-n+i})}{\sinh(\mu_{k+j} - \mu_{k+i})} \\
&\quad \times \prod_{i=1}^{N-n} \prod_{j=1}^{n} \frac{\sinh(\lambda_i - \lambda_{N-n+j})}{\sinh(\lambda_i - \mu_{k+j} - \eta/2)} \\
&\quad \times \prod_{i=1}^{N} \prod_{j=1}^{n} \frac{\sinh(\lambda_i - \mu_{k+j} + \eta/2)}{\sinh(\lambda_i - \lambda_{N-n+j} + \eta)} \frac{\det \psi'(\{\lambda_i\}, \{\mu_{k+j}\})}{\det \varphi'(\{\lambda_i\})}.
\end{aligned} \tag{27}$$

The first $N - n$ rows of the $N \times N$ matrix ψ' are equal to those of the matrix φ', but the other n rows are different:

$$\psi'_{ij} = \varphi'_{ij} \qquad 1 \leqslant i \leqslant N - n \tag{28}$$

$$\psi'_{ij} = \frac{\sinh \eta}{\sinh(\lambda_j - \mu_{k+i} - \eta/2)\,\sinh(\lambda_j - \mu_{k+i} + \eta/2)} \qquad N - n + 1 \leqslant i \leqslant N. \tag{29}$$

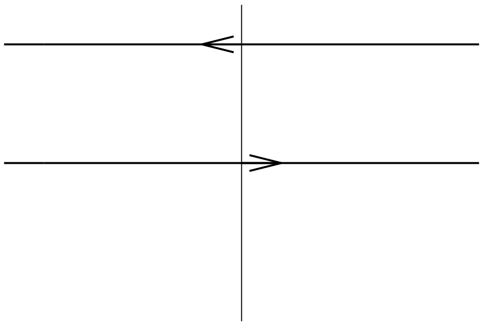

Figure 1. The contour $\mathcal{C}$ in the complex plane.

Finally, we can rewrite the ratio of the determinants in (27) as one determinant by inverting φ':

$$\frac{\det \psi'}{\det \varphi'} = \det(\psi' \varphi'^{-1}). \tag{30}$$

To proceed we have to calculate the matrix $\psi'\varphi'^{-1}$. The first $N - n$ rows of this matrix can be easily calculated using (28)

$$(\psi'\varphi'^{-1})_{ij} = \delta_{ij} \qquad 1 \leqslant i \leqslant N - n. \tag{31}$$

In section 4 we will calculate the other rows in the limit ($M \to \infty$) for a particular set of solutions of the Bethe Ansatz equations.

4. Thermodynamic limit

In this section we will describe the thermodynamic limit $M \to \infty$. To be able to take this limit we need some information on the distribution of the solutions of the Bethe Ansatz equations. The solutions of (13) fall into two classes depending on the value of η. These are the so-called massive regime, where $\Delta = \cosh \eta > 1$ and the massless regime where $|\Delta| \leqslant 1$. In this paper we will concentrate on the massless case only. Since $|\Delta| \leqslant 1$ we will use the parameterization $\gamma = \mathrm{i}\eta$ and furthermore, we will restrict our attention to the interval $\pi/2 > \gamma \geqslant 0$. We will consider the class of solutions of (13) for which the imaginary part of each λ_j is either 0 or $\pi/2$. These are the so-called 1-strings in the language of [19]. In the limit $M \to \infty$, the solutions we consider thus belong to a directed contour $\mathcal{C}$ (figure 1) which is defined by

$$\mathcal{C} = (-\infty, \infty) \cup (\infty + \mathrm{i}\pi/2, -\infty + \mathrm{i}\pi/2). \tag{32}$$

Now we will derive the logarithmic form of the Bethe Ansatz equations in a more precise manner than was done in (14). For this purpose we define the function p_n by

$$p_n(\lambda) = \begin{cases} 2\,\arctan(\tanh\lambda \cot n\gamma/2) & \text{for} \quad \mathrm{Im}\,\lambda = 0 \\ -2\,\arctan(\coth\lambda \tan n\gamma/2) & \text{for} \quad \mathrm{Im}\,\lambda = \pi/2. \end{cases} \tag{33}$$

For $\sin n\gamma > 0$ this function is monotonously increasing (decreasing) on the line $\mathrm{Im}\,\lambda = 0$ ($\mathrm{Im}\,\lambda = \pi/2$). The logarithmic version of the Bethe Ansatz equations can then be written as

$$\phi(\lambda_i) = 2\pi n_i \tag{34}$$

where the function ϕ is given by

$$\phi(\lambda) = \sum_{k=1}^{M} p_1(\lambda - \mu_k) - \sum_{j=1}^{N} p_2(\lambda - \lambda_j). \tag{35}$$

The numbers n_i appearing on the right-hand side of (34) are integers for N odd and half integers for N even. For every (half) integer $\{n_i\}$ there are two solutions of the Bethe Ansatz equations, corresponding to the two different values of the imaginary part. A solution thus is uniquely specified by a set of integers and a corresponding set of parities, where the parity of a solution is defined by

$$v = 1 - \frac{4}{\pi}\,\mathrm{Im}\,\lambda. \tag{36}$$

Given a set of (half) integers $\{n_i\}$ and a set of parities $\{v_i\}$, a solution λ_j of (34) is called a particle. A solution $\lambda_{\mathrm{h}} = \underline{\lambda}_{\mathrm{h}} + \mathrm{i}\pi(1 - v_{\mathrm{h}})/4$ to the equation

$$\phi(\lambda_{\mathrm{h}}) = 2\pi m \qquad m \notin \{n_i\} \quad \text{or} \quad v_{\mathrm{h}} \notin \{v_i\} \tag{37}$$

is called a hole. In the thermodynamic limit, the particles and holes have finite distribution densities ρ_{p} and ρ_{h}, defined by

$$M\rho_{\mathrm{p}}(\lambda)\mathrm{d}\lambda \quad \text{number of particles in } [\lambda, \lambda + \mathrm{d}\lambda] \tag{38}$$

$$M\rho_{\mathrm{h}}(\lambda)\mathrm{d}\lambda \quad \text{number holes in } [\lambda, \lambda + \mathrm{d}\lambda]. \tag{39}$$

Note that the 1-form $\mathrm{d}\lambda$ has a direction corresponding to that of $\mathcal{C}$. The density ρ_{tot} of the total possible solutions, or vacancies, is given by

$$\rho_{\mathrm{tot}}(\lambda) = \rho_{\mathrm{p}}(\lambda) + \rho_{\mathrm{h}}(\lambda). \tag{40}$$

Since we are dealing with an inhomogeneous model, it will be useful to define the densities $\tilde{\rho}_{\mathrm{tot}}$ by

$$\rho_{\mathrm{tot}}(\lambda) = \frac{1}{M}\sum_{k=1}^{M} \tilde{\rho}_{\mathrm{tot}}(\lambda - \mu_k). \tag{41}$$

The corresponding particle and hole densities are given by

$$\tilde{\rho}_{\mathrm{p}}(\lambda - \mu_k) = \vartheta(\lambda)\tilde{\rho}_{\mathrm{tot}}(\lambda - \mu_k) \tag{42}$$

$$\tilde{\rho}_{\mathrm{h}}(\lambda - \mu_k) = (1 - \vartheta(\lambda))\tilde{\rho}_{\mathrm{tot}}(\lambda - \mu_k) \tag{43}$$

where the Fermi weight $\vartheta(\lambda)$ is given by

$$\vartheta(\lambda) = \frac{\rho_{\mathrm{p}}(\lambda)}{\rho_{\mathrm{tot}}(\lambda)}. \tag{44}$$

Using these densities we can take the limit $M \to \infty$. It follows that (34) in the thermodynamic limit can be written as

$$\lim_{M\to\infty} \frac{1}{M}\phi(\lambda) = \pi\left(-1 + 2\int_{-\infty}^{\lambda} \rho_{\mathrm{tot}}(\lambda')\,\mathrm{d}\lambda'\right) \tag{45}$$

where the integration is along the contour $\mathcal{C}$, (figure 1). Differentiating with respect to λ we find the formula

$$\rho_{\mathrm{tot}}(\lambda) = K_1^{\mathrm{tot}}(\lambda) - \int_{\mathcal{C}} K_2(\lambda - \lambda')\rho_{\mathrm{p}}(\lambda')\,\mathrm{d}\lambda'. \tag{46}$$

The function K_1^{tot} is defined by

$$K_1^{\mathrm{tot}}(\lambda) = \lim_{M\to\infty} \frac{1}{M}\sum_{k=1}^{M} K_1(\lambda - \mu_k) \tag{47}$$

and K_n is given by

$$K_n(\lambda) = \frac{1}{2\pi} p'_n(\lambda) = \frac{1}{2\pi} \frac{\sin n\gamma}{\sinh(\lambda - \mathrm{i}n\gamma/2)\sinh(\lambda + \mathrm{i}n\gamma/2)}. \tag{48}$$

The thermodynamic limit of φ' is found using (45) and the fact that $\phi(\lambda) = \varphi(\lambda) \bmod \pi$:

$$\varphi'_{ij} = -2\pi\mathrm{i}(M\delta_{ij}\rho_{\mathrm{tot}}(\lambda_i) + K_2(\lambda_i - \lambda_j)). \tag{49}$$

Now we are in a postion to return to the calculation at the end of section 3. Remember that we want to calculate the last n rows of the matrix $\psi'\varphi'^{-1}$. For this purpose we recall from (29) that

$$\begin{aligned}\psi'_{N-n+i,j} &= -2\pi\mathrm{i}K_1(\lambda_j - \mu_{k+i})\\ &= -2\pi\mathrm{i}\left(\tilde{\rho}_{\mathrm{tot}}(\lambda_j - \mu_{k+i}) + \int_{\mathcal{C}} K_2(\lambda_j - \lambda')\vartheta(\lambda')\tilde{\rho}_{\mathrm{tot}}(\lambda' - \mu_{k+i})\,\mathrm{d}\lambda'\right)\end{aligned} \tag{50}$$

where in the second line we have used (42) and (46). From (49), however, and the fact that K_2 is symmetric, it follows that this is precisely equal to

$$\psi'_{N-n+i,j} = \frac{1}{M}\sum_{l=1}^{N} \frac{\tilde{\rho}_{\mathrm{tot}}(\lambda_l - \mu_{k+i})}{\rho_{\mathrm{tot}}(\lambda_l)}\varphi'_{lj}. \tag{51}$$

We thus conclude that

$$(\psi'\varphi'^{-1})_{ij} = \delta_{ij} \qquad 1 \leqslant i \leqslant N-n \tag{52}$$

$$(\psi'\varphi'^{-1})_{N-n+i,j} = \frac{\tilde{\rho}_{\mathrm{tot}}(\lambda_j - \mu_{k+i})}{M\rho_{\mathrm{tot}}(\lambda_j)} \qquad 1 \leqslant i \leqslant n. \tag{53}$$

The determinant of this matrix can be written concisely as

$$\det(\psi'\varphi'^{-1}) = \det\tilde{S}\frac{1}{M^n}\prod_{j=1}^{n}\rho_{\mathrm{tot}}^{-1}(\lambda_{N-n+j}) \tag{54}$$

where the $n \times n$ matrix $\tilde{S}$ is given by

$$\tilde{S}_{ij} = \tilde{\rho}_{\mathrm{tot}}(\lambda_{N-n+j} - \mu_{k+i}). \tag{55}$$

Finally, using (22), (24), (27) and (54), the emptiness formation probability can be written as

$$\begin{aligned}\langle\pi_{k+1}\dots\pi_{k+n}\rangle &= \frac{1}{M^n\prod_{l<m}\sinh(\mu_{k+l} - \mu_{k+m})}\\ &\quad\times\sum_{i_1=1}^{N}\dots\sum_{i_n=1}^{N} H\left(\{\lambda_{i_l}\},\{\mu_{k+l}\}\right)\prod_{l=1}^{n}\rho_{\mathrm{tot}}^{-1}\left(\lambda_{i_l}\right)\end{aligned} \tag{56}$$

where the function H is given by

$$\begin{aligned}H\left(\{\lambda_{i_l}\},\{\mu_{k+l}\}\right) &= \frac{\det\tilde{S}\left(\{\lambda_{i_l}\},\{\mu_{k+l}\}\right)}{\prod_{l<m}\sinh\left(\lambda_{i_m} - \lambda_{i_l} - \mathrm{i}\gamma\right)}\\ &\quad\times\prod_{l=1}^{n}\left(\prod_{m=1}^{l-1}\sinh\left(\lambda_{i_l} - \mu_{k+m} - \mathrm{i}\gamma/2\right)\prod_{m=l+1}^{n}\sinh\left(\lambda_{i_l} - \mu_{k+m} + \mathrm{i}\gamma/2\right)\right).\end{aligned} \tag{57}$$

The last step in deriving an expression for the emptiness formation probability in the thermodynamic limit is to replace the sums in (56) by integrals using the above discussion. We then arrive at the following multiple integral expression:

$$\langle \pi_{k+1} \ldots \pi_{k+n} \rangle = \frac{1}{\prod_{l<m} \sinh(\mu_{k+l} - \mu_{k+m})} \int_{\mathcal{C}} \ldots \int_{\mathcal{C}} H(\{\lambda_l\}, \{\mu_{k+l}\}) \prod_{l=1}^{n} \vartheta(\lambda_l)\, \mathrm{d}\lambda_l \qquad (58)$$

where the Fermi weight $\vartheta(\lambda)$ is defined in (44). We remind the reader that the integration is along the directed contour $\mathcal{C}$ (equation (32), figure 1).

5. Conclusion

In this paper we have obtained a multiple integral expression for the emptiness formation probability (EPF) on the central horizontal line of the inhomogeneous six-vertex model with domain wall boundaries. We derived this expression in the thermodynamic limit when the inhomogeneities are chosen from a particular set of solutions of the Bethe Ansatz equations, namely those without a bound state but otherwise arbitrary. This result is a first step to obtaining an expression for the EPF for general solutions of the Bethe Ansatz equations, i.e. also for bound state solutions. We expect that certain properties of the EPF are independent of the special choice of inhomogeneities. Ultimately we hope to learn more about such properties by studying the EPF averaged over Bethe Ansatz solutions.

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